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ON CONTACT PROBLEMS FOR WEDGE-SHAPED PLATES*

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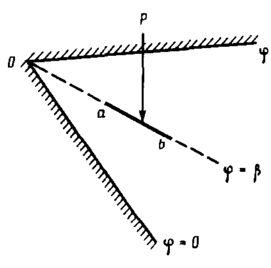
Problems concerning the pressing of a thin linear inclusion in the form of a stiffness rib into a plate having a wedge-shape in planform are studied. The plate edges are either rigidly clamped or simply supported. The integral equations (IE) of these problems have a kernel symbol of the type tanh or coth, where their solution is sought in the class of functions with non-integrable singularities /1/. New simple special approximations are proposed for the kernel symbols that will enable the problem IE to be reduced to IE in two regions solvable in closed form. Analogous approximations of the tanh or coth of the kernel symbols of IE whose solutions have integrable singularities were examined in /2/, for example. Here IE in two regions also occurred. Since the approximations of the tanh or coth do not always enable a solution to be obtained with satisfactory accuracy, a complicated approximation of the kernel symbol with an arbitrary given error is introduced. The IE solution is also constructed in closed form for such an approximation. Examples are presented when an exact solution of the problem is obtained successfully. The results may be useful in the analysis of structural elements in construction.

1. Let a rigid stiff inclusion of magnitude $f(r)$ (see the figure) be impressed by a force P into a Kirchhoff-Love plate of wedge-shaped planform with apex angle α along the segment $a \leqslant r \leqslant b$ of the ray $\varphi = \beta$. It is assumed that the plate edge is a) rigidly clamped; b) simply supported; c) the edge $\varphi = 0$ is simply supported, the edge $\varphi = \alpha$ is rigidly clamped. It is required to determine the contact forces $\varphi(r)$, $a \leqslant r \leqslant b$ equal to the jump of the generalized transverse forces on the inclusion, and also to find the magnitude of the force P and moment M applied to the inclusion for a given $f(r)$. A method of obtaining Green's function for wedge-shaped plates with the boundary conditions under consideration here is indicated in /3/, based on application of the Mellin integral transform. After having found Green's functions of three boundary-value problems and having introduced dimensionless quantities by means of the formulas

$$(r/b)^R = r^*, \quad (\rho/b)^R = \rho^*, \quad k = (a/b)^R \tag{1.1}$$
$$\varphi(r)\, r^2 = RDr^{*2}\varphi^*(r^*), \quad 4f(r)\, r^{-1} = R^{-2} f^*(r^*)\, r^{*-1}$$



(D is the plate stiffness and R is a positive constant), the IE can be written in the form (we later omit the asterisks $\gamma = \alpha - \beta$, $\delta = \alpha + \beta$):

$$R^2 \int_k^1 \rho\varphi(\rho)\, K\left(\ln\left(\frac{\rho}{r}\right) R^{-1}\right) d\rho = \frac{\pi f(r)}{r}, \quad k \leqslant r \leqslant 1 \tag{1.2}$$

$$K(t) = \int_0^\infty L(u) \frac{\cos ut\, du}{(1+u^2)\, u} + \frac{\pi}{2} Bt$$

a) $B = 0$, $L(u) = 2\{[u \operatorname{ch} \gamma u \sin \gamma - \operatorname{sh} \gamma u \cos \gamma]\, [u^2 \operatorname{ch} \gamma u \times \sin \alpha \sin \beta +$
$u(\operatorname{sh} \gamma u \sin \delta - \operatorname{sh} \delta u \sin \gamma)/2 - \operatorname{sh} \alpha u \operatorname{sh} \beta u \cos \gamma] - (1+u^2) \times$
$\operatorname{sh} \gamma u \sin \gamma\, [u \operatorname{sh} \gamma u \sin \alpha \sin \beta - \operatorname{sh} \alpha u \operatorname{sh} \beta u \sin \gamma]\} \times (\operatorname{sh}^2 \alpha u - u^2 \sin^2 \alpha)^{-1}$

b) $B = \lim uL(u)$, $u \to 0$ for $\alpha = \pi$; $\alpha = 2\pi$ and $\beta \neq \pi$; in the remaining cases $B = 0$;

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$L(u) = 2\{u\,(\mathrm{ch}\,\alpha u\,\mathrm{sh}\,\beta u\,\mathrm{sh}\,\gamma u\sin\alpha\cos\beta\cos\gamma - \mathrm{ch}\,\alpha u\,\mathrm{ch}\,\beta u\,\mathrm{ch}\,\gamma u\sin\alpha\cdot\sin\beta\sin\gamma - \mathrm{sh}\,\alpha u\,\mathrm{ch}\,\beta u\,\mathrm{sh}\,\gamma u\cos\alpha\sin\beta\cos\gamma - \mathrm{sh}\,\alpha u\,\mathrm{sh}\,\beta u\,\mathrm{ch}\,\gamma u\cos\alpha\cdot\cos\beta\sin\gamma) + \mathrm{sh}\,\alpha u\,\mathrm{sh}\,\beta u\,\mathrm{sh}\,\gamma u\cos\alpha\cos\beta\cos\gamma - \mathrm{sh}\,\alpha u\,\mathrm{ch}\,\beta u\,\mathrm{ch}\,\gamma u\cos\alpha\cdot\sin\beta\sin\gamma + \mathrm{ch}\,\alpha u\,\mathrm{ch}\,\beta u\,\mathrm{sh}\,\gamma u\sin\alpha\sin\beta\cos\gamma + \mathrm{ch}\,\alpha u\,\mathrm{sh}\,\beta u\,\mathrm{ch}\,\gamma u\sin\alpha\cdot\cos\beta\sin\gamma\}\,(\mathrm{ch}^2\alpha u - \cos^2\alpha)^{-1}$

c) $B = 0,\quad L(u) = 2\{\mathrm{sh}\,\gamma u\,(\mathrm{sh}\,\delta u\sin\gamma - \mathrm{sh}\,\gamma u\sin\delta)(\mathrm{ch}\,2\alpha u\sin\gamma + \sin\delta) + \mathrm{sh}\,2\alpha u\,\mathrm{sh}\,\gamma u\cos\gamma\,(\mathrm{ch}\,\delta u\cos\gamma - \mathrm{ch}\,\gamma u\cos\delta) + u\,[(\mathrm{sh}\,\delta u\sin\gamma - \mathrm{sh}\,\gamma u\sin\delta)(\mathrm{sh}\,2\alpha u\,\mathrm{sh}\,\gamma u\cos\gamma - \mathrm{ch}\,\gamma u\sin 2\alpha\sin\gamma) - (\mathrm{ch}\,\delta u\cos\gamma - \mathrm{ch}\,\gamma u\cos\delta)(\mathrm{sh}\,2\alpha u\,\mathrm{ch}\,\gamma u\sin\gamma + \mathrm{sh}\,\gamma u\sin 2\alpha\cos\gamma)] + u^2[(\mathrm{ch}\,\delta u\cos\gamma - \mathrm{ch}\,\gamma u\cos\delta)\,\mathrm{ch}\,\gamma u\sin 2\alpha\sin\gamma - (\mathrm{sh}\,\delta u\sin\gamma - \mathrm{sh}\,\gamma u\sin\delta)(\mathrm{sh}\,\gamma u\cos 2\alpha + \mathrm{sh}\,\delta u)\sin\gamma]\} \times [(\mathrm{ch}\,2\alpha u - \cos 2\alpha)(\mathrm{sh}\,2\alpha u - u\sin 2\alpha)]^{-1}$

Note that the function $L(u)$ is a function of the type tanh u for $B = 0$ or coth u when $B \neq 0$ in its asymptotic properties as $u \to 0,\ u \to \infty$ and is the residue of the function $L(u)$ at zero. For $L(u)$ of the coth u type, the integral in the kernel $K(t)$ is understood in the sense of the theory of generalized functions exactly, say, as the integral in the kernel of the IE of a plane contact problem for an elastic wedge with a hinge-supported lower face /2/. It follows /1/ from the asymptotic behaviour of $L(u)$ as $u \to \infty$ that the solution of the IE (1.2) must be sought in the class of functions with non-integrable singularities of order $-^3/_2$ at the point $\rho = k,\ 1$.

2. We introduce the following approximations for $L(u),\ 0 < u < \infty$ of the tanh u and coth u types, respectively $(R = \pi/(2A))$:

$$\frac{L(u)}{1+u^2} \approx \frac{\mathrm{th}\,Au}{u^2+R^2},\quad \frac{A}{R^2} = \lim_{u\to 0}\frac{L(u)}{u} \tag{2.1}$$
$$\frac{L(u)}{1+u^2} \approx \frac{\mathrm{cth}\,Au}{u^2+R^2},\quad \frac{1}{AR^2} = B$$

Using the known intregrals /4/, it can be found that

$$\int_0^\infty \frac{\mathrm{th}\,Au\cos ut}{u\,(u^2+R^2)}\,du = \frac{1}{R^2}\left\{\ln\mathrm{cth}\frac{Rt}{2} + \mathrm{ch}\,Rt\ln(2\,\mathrm{sh}\,Rt) - RT\,\mathrm{sh}\,Rt\right\} \tag{2.2}$$
$$\int_0^\infty \frac{\mathrm{cth}\,Au\cos ut}{u\,(u^2+R^2)}\,du = -\frac{1}{R^2}\left\{\mathrm{ch}\,Rt\ln\mathrm{cth}\frac{Rt}{2} + \ln(2\,\mathrm{sh}\,Rt)\right\}$$

Inserting (2.1) and (2.2) into the IE (1.2), we will have two IE for the cases of kernel symbols of the type tanh u and coth u $(k \leqslant r \leqslant 1)$

$$\int_k^1 \varphi(\rho)\left\{\frac{(\rho-r)^2}{2}\ln|\rho-r| + \frac{(\rho+r)^2}{2}\ln(\rho+r) - r^2\ln r - \rho^2\ln\rho\right\}d\rho = \pi f(r) \tag{2.3}$$

$$\int_k^1 \varphi(\rho)\left\{\frac{(\rho-r)^2}{2}\ln|\rho-r| - \frac{(\rho+r)^2}{2}\ln(\rho+r) + 2r\rho\ln\rho\right\}d\rho = \pi f(r) \tag{2.4}$$

The solution of the IE (2.3) should obviously be sought as an even solution of the corresponding IE in two regions while the solution of the IE (2.4) should be sought as odd. We will make a regularizing substitution of the desired functions in (2.3) and (2.4), respectively, by the formulas $(g(r) = \sqrt{(1-r^2)(r^2-k^2)})$:

$$\varphi(\rho) = \varphi_+(\rho) + (\alpha_+\rho + \beta_+\rho^3)\,g^{-3}(\rho) \tag{2.5}$$
$$\varphi(\rho) = \varphi_-(\rho) + (\alpha_- + \beta_-\rho^2)\,g^{-3}(\rho) \tag{2.6}$$

where $\alpha_\pm$ and $\beta_\pm$ are selected in such a manner that $\varphi_\pm(\rho)$ have integrable singularities at the points $\rho = k, 1$. We substitute (2.5) into (2.3), differentiate the equation obtained three times with respect to r assuming that $f(r) \in H_3^\sigma(k, 1),\ 0 < \sigma \leqslant 1$ and, dividing by r, obtain the known singular IE

$$\int_k^1 \varphi_+(\rho)\frac{2d\rho}{\rho^2-r^2} = -\pi\frac{f'''(r)}{r} - \frac{2P_+}{r^2} \quad (k \leqslant r \leqslant 1), \quad P_+ = \int_k^1 \varphi_+(\rho)\,d\rho \tag{2.7}$$

in two regions. Its solution has the form

$$\varphi_+(\rho) = \frac{2\rho}{\pi g(\rho)}\left\{P_+\frac{k}{\rho^2} + \int_k^1 \frac{g(r)f'''(r)}{r^2-\rho^2}\,dr\right\} \tag{2.8}$$

It is taken into account in (2.8) that /4/

$$\int_k^1 \frac{g(r)\,dr}{r(r^2-\rho^2)} = \frac{\pi}{2}\left(\frac{k}{\rho^2}-1\right) \quad (k \leqslant \rho \leqslant 1)$$

Similarly, substituting (2.6) into (2.4) and differentiating thrice with respect to r, we find

$$\varphi_-(\rho) = \frac{2}{\pi g(\rho)}\left\{M_- + \int_k^1 \frac{g(r)f'''(r)\,r}{r^2-\rho^2}\,dr\right\}, \quad M_- = \int_k^1 \rho\varphi_-(\rho)\,d\rho \tag{2.9}$$

Three unknown constants $\alpha_\pm, \beta_\pm, P_+, M_-$ are present in (2.5), (2.8) and (2.6), (2.9). To obtain a system of three algebraic equations in α_+, β_+, P_+ we operate on the IE (2.3) in which the function $\varphi(\rho)$ is replaced by (2.5) and (2.8), with the operators

$$\frac{rdr}{g(r)}\int_k^1, \quad \frac{rdr}{g^3(r)}\int_k^1, \quad \frac{r^3dr}{g^3(r)}\int_k^1$$

After taking quadratures we will have the system of equations

$$\begin{gathered}
\int_k^1 \frac{rf(r)\,dr}{g(r)} + \left(\frac{1}{4}\ln\frac{1-k^2}{4}+1\right)\int_k^1 g(r)f'''(r)\,dr + \int_k^1 \frac{\rho^2(\rho\ln\rho-2d_0)}{\pi g(\rho)}\,d\rho\,\times \\
\int_k^1 \frac{g(r)f'''(r)}{r^2-\rho^2}\,dr = \alpha_+\left(d_0b_1-\frac{c_1}{2}\right) + \beta_+\left(d_0b_2-\frac{\pi}{8}\ln\frac{1-k^2}{4} - \right. \\
\left.\frac{\pi}{2}-\frac{c_2}{2}\right) + P_+\left(\frac{d_0}{\pi}2kK' + \frac{1+6k+k^2}{8} + \frac{(1+k)^2}{8}\ln\frac{1-k}{1+k}\right) \\
\int_k^1 \frac{rf(r)}{g^3(r)}\,dr - \frac{1}{(1-k^2)^2}\int_k^1 g(r)f'''(r)\,dr + \int_k^1 \frac{2a_1\rho^2d\rho}{\pi g(\rho)}\int_k^1 \frac{g(r)f'''(r)}{r^2-\rho^2}\,dr = \\
-\alpha_+a_1b_1 + \beta_+\left(\frac{\pi}{2(1-k^2)^2}-a_1b_2\right) + P_+\left(\frac{k^4-2k-1}{2(1-k^2)^2} - \frac{2ka_1}{\pi}K' - \frac{c_1}{\pi}\right) \\
\int_k^1 \frac{r^3f(r)\,dr}{g^3(r)} - d_1\int_k^1 g(r)f'''(r)\,dr + \int_k^1 \frac{2a_2\rho^2d\rho}{\pi g(\rho)}\int_k^1 \frac{g(r)f'''(r)}{r^2-\rho^2}\,dr = \\
-\alpha_+a_2b_1 + \beta_+\left(\frac{\pi}{2}d_1-a_2b_2\right) - P_+\left(kd_1+\frac{2ka_2}{\pi}K'-d_2+\frac{c_2}{\pi}\right)
\end{gathered} \tag{2.10}$$

where the following notation has been introduced:

$$a_n = \frac{1}{\pi}\int_k^1 t^{2n}\ln\left|\frac{t-\xi}{t+\xi}\right|\frac{dt}{g^3(t)} \quad (k \leqslant \xi \leqslant 1) \tag{2.11}$$

$$b_n = \int_k^1 \frac{t^{2n}}{g^3(t)}\,dt, \quad c_n = \int_k^1 \frac{t^{2n+1}\ln t}{g^3(t)}\,dt, \quad n = 0, 1, 2$$

$$d_0 = K - E - \frac{1}{4}, \quad d_1 = \frac{1+k^2}{2(1-k^2)^2}, \quad d_2 = \frac{1}{4}\ln\frac{4}{1-k^2} - \frac{1+k^4}{2(1-k^2)^2}$$

and $K = K(k)$, $K' = K'(k)$, $E = E(k)$ and $E' = E'(k)$ and complete elliptic integrals. The integrals with non-integrable singularities of order $-^3/_2$ in (2.10) and (2.11) are understood in the finite-part sense /5/. The values of the integrals a_n, b_n, c_n $(n = 0, 1, 2)$ can be calculated as a function of k by using a computer.

Similarly, after acting on the IE (2.4) with the operators

$$\frac{dr}{g(r)}\int_k^1, \quad \frac{dr}{g^3(r)}\int_k^1, \quad \frac{r^2dr}{g^3(r)}\int_k^1$$

and taking (2.6) and (2.9) into account, we obtain a system of three linear equation in α_-, β_-, M_-

$$\int_k^1 \frac{f(r)\,dr}{g(r)} + \int_k^1 \frac{(K - E - 2\rho\ln\rho + K\rho^2)}{\pi g(\rho)}\,d\rho \int_k^1 \frac{g(r)\,f'''(r)\,r}{r^2-\rho^2}\,dr = \tag{2.12}$$

$$\alpha_-\left(\frac{E-K}{2}b_0 - \frac{K}{2}b_1 + c_0\right) + \beta_-\left(\frac{E-K}{2}b_1 - \frac{K}{2}b_2 + c_1\right) +$$

$$M_-\left(\frac{E-K}{\pi}K' - \frac{K}{\pi}E' + \frac{1}{2}\left(\ln\frac{4}{1-k^2} - 1\right) + \ln\frac{1+k}{2}\right)$$

$$\int_k^1 \frac{f(r)\,dr}{g^3(r)} - \int_k^1 \frac{(a_1 + a_0\rho^2)\,d\rho}{\pi g(\rho)} \int_k^1 \frac{g(r)\,f'''(r)\,r}{r^2-\rho^2}\,dr =$$

$$\alpha_-\left(\frac{a_1b_0}{2} + \frac{a_0b_1}{2}\right) + \beta_-\left(\frac{a_1b_1}{2} + \frac{a_0b_2}{2}\right) + M_-\left(\frac{a_1}{\pi}K' + \frac{a_0}{\pi}E' + \frac{2}{(1-k^2)^2}\right)$$

$$\int_k^1 \frac{r^2 f(r)\,dr}{g^3(r)} - \int_k^1 \frac{(a_2 + a_1\rho^2)\,d\rho}{\pi g(\rho)} \int_k^1 \frac{g(r)\,f'''(r)\,r}{r^2-\rho^2}\,dr =$$

$$\alpha_-\left(\frac{a_2b_0}{2} + \frac{a_1b_1}{2}\right) + \beta_-\left(\frac{a_2b_1}{2} + \frac{a_1b_2}{2}\right) + M_-\left(\frac{a_2}{\pi}K' + \frac{a_1}{\pi}E' + 2d_1\right)$$

After determining $\alpha_\pm, \beta_\pm, P_+, M_-$ from (2.10) and (2.12), the closed solution of the IE (2.3) and (2.4) are found from (2.5), (2.8) and (2.6), (2.9).

We will present some examples of specific approximations of the form (2.1). In the case of problem a), for $\alpha = 2\beta = \pi$, $A = 1.321$ the error of the approximation is $\varkappa = 10\%$ while in the case of the problem b) for $\alpha = 2\beta = \pi/2$, $A = 1.121$ the error is $\varkappa = 5\%$. For case b) for $\alpha = 2\beta = \pi$ $L(u) = \operatorname{cth} \pi u/2$, $A = \pi/2$ the problem has indeed as exact solution that remains valid with a certain error even in a small neighbourhood of the value of the angle $\beta = \pi/2$.

3. Approximations of the form (2.1) have satisfactory errors but not for all values of the angles α, β $(0 < \alpha \leqslant 2\pi,\ 0 < \beta < \alpha)$. A more complex approximation has been proposed* of the form

$$\frac{L(u)}{1+u^2} \approx \frac{\operatorname{th} Au}{u^2+\gamma_0^2}\prod_{n=1}^{N}\frac{u^2+\delta_n^2}{u^2+\gamma_n^2} \tag{3.1}$$

for the case of the IE kernel symbol of the type tanh u. The constants $\gamma_0, \delta_n, \gamma_n$ $(n = 1, \ldots, N)$ are chosen from the condition for best approximation of the form (3.1) along the real axis. By increasing N any accuracy of the approximation can be achieved.

In the case of the approximation (3.1) we introduce instead of (1.1) non-dimensional quantities according to the formulas

$$x = \lambda\ln(r/a) - 1,\ \xi = \lambda\ln(\rho/a) - 1,\ \lambda = 2\,[\ln(b/a)]^{-1} \tag{3.2}$$
$$4\lambda^3 f(r)/r = G(x),\ \ \varphi(r)\,r^2D^{-1} = \psi(x)$$

Then the IE of the problem takes the form

$$\lambda^2\int_{-1}^{1}\psi(\xi)\,K\left(\frac{\xi - x}{\lambda}\right)d\xi = \pi G(x), \quad |x| \leqslant 1 \tag{3.3}$$

We make the change of variable $u = \lambda u'$ and introduce the notation $\delta_n = \lambda\delta_n'$, $\gamma_n = \lambda\gamma_n'$, $\gamma_0 = \lambda\gamma_0'$ in the integral in expression (1.2) for $K(t)$. After this the IE (3.3) can be rewritten taking (3.1) into account as (the primes are omitted)

*Zelentsov V.B., Asymptotic Methods of Solving Mixed Problems of the Theory of Thin Plate Bending. Candidate Dissertation, Rostov Univ., Rostov on/Don 1984.

$$\int_{-1}^{1} \psi(\xi) K(\xi - x)\, d\xi = 2\pi G(x), \quad |x| \leqslant 1 \tag{3.4}$$

$$K(t) = \int_{-\infty}^{\infty} \frac{\operatorname{th}(A\lambda u)}{u(u^2+\gamma_0^2)} \prod_{n=1}^{N} \frac{u^2+\delta_n^2}{u^2+\gamma_n^2} e^{-iut} du \tag{3.5}$$

We apply the method developed in /6/ to solve the IE (3.4) with kernel (3.5). For simplicity we assume that $G(x)$ is an even function and $G(x) = \operatorname{ch} \varepsilon x$, ε is a complex constant. We represent the IE (3.4) and (3.5) in the form of an ordinary differential equation

$$\mathrm{P}_1(L)\,\Omega(x) = 2\pi \mathrm{P}_2(-\varepsilon^2) \operatorname{ch} \varepsilon x, \quad |x| \leqslant 1 \tag{3.6}$$

$$\Omega(x) = \int_{-1}^{1} \psi(\xi)\, d\xi \int_{-\infty}^{\infty} \frac{\operatorname{th}(A\lambda u)}{u} \cos u(x-\xi)\, du \tag{3.7}$$

$$L = -\frac{d^2}{dx^2}, \quad \mathrm{P}_1(u^2) = \prod_{n=1}^{N} (u^2+\delta_n^2), \quad \mathrm{P}_2(u^2) = (u^2+\gamma_0^2) \prod_{n=1}^{N} (u^2+\gamma_n^2)$$

Inverting (3.6) and using (3.7), we obtain an equation to determine $\psi(x)$

$$\int_{-1}^{1} \psi(\xi)\, d\xi \int_{-\infty}^{\infty} \frac{\operatorname{th}(A\lambda u)}{u} \cos u(x-\xi)\, du = \tag{3.8}$$

$$2\pi \frac{\mathrm{P}_2(-\varepsilon^2)}{\mathrm{P}_1(-\varepsilon^2)} \operatorname{ch} \varepsilon x + 2\pi \sum_{n=1}^{N} C_n \operatorname{ch} \delta_n x, \quad |x| \leqslant 1$$

Here C_n are unknown constants.

Since the solution of the IE (3.4) is sought in the class of functions with non-integrable singularities of order $-{}^3/_2$, we regularize the outer integral in (3.8) by introducing the function

$$\psi^*(x) = \psi(x) - C(\operatorname{ch}\theta - \operatorname{ch}\theta x)^{-3/2}, \quad \theta = \pi/A\lambda \tag{3.9}$$

where $\psi^*(x)$ belongs to the class of functions with integrable singularities of order $-{}^1/_2$ at the points $|x| = 1$. Taking account of the results in /6/, we obtain for the Fourier transform $\Psi(\beta)$ of the function $\psi(x)$ (understanding the corresponding integral in the finite-part sense)

$$\Psi(\beta) = 2\pi C \frac{\sqrt{2}}{\theta \operatorname{sh}\theta} \frac{\lambda^2}{Q_{-1/2}} R(i\beta, 0) + \Phi_+(\beta) \tag{3.10}$$

$$\Psi(\beta) = \int_{-1}^{1} \psi(\xi) \cos \beta\xi\, d\xi, \quad Q_\nu^\mu = Q_\nu^\mu(\operatorname{ch}\theta)$$

The functions $\Phi_+(\beta)$, $R(u, v)$ are given by (2.6) from /6/. The solution itself has the form

$$\psi(x) = \frac{\theta}{2\pi} C_0 \frac{\operatorname{sh}\theta Q_{-1/2} + 2(\operatorname{ch}\theta - \operatorname{ch}\theta x) Q^1_{-1/2}}{\theta_{-1/2} \sqrt{2(\operatorname{ch}\theta - \operatorname{ch}\theta x)^3}} + \varphi_+(x), \quad C_0 = \frac{2\pi C\sqrt{2}}{\theta \operatorname{sh}\theta} \tag{3.11}$$

where $\varphi_+(x)$ is given by (2.9) in /6/. The constants $C_0, C_1, \ldots, C_N$ in (3.11) are found from the linear algebraic system

$$\sum_{n=0}^{N} x_n = f_m + \sum_{n=0}^{N} a_{mn} x_n, \quad m = 0, 1, \ldots, N \tag{3.12}$$

$$\left(x_0 = -C_0 Q^1_{-1/2}, \quad a_{m0} = -\frac{Q_{-1/2} Q^1_{-1/2+\gamma_m/\theta}}{Q^1_{-1/2} Q_{-1/2+\gamma_m/\theta}} \right)$$

where the quantities x_n, a_{mn}, f_m $(n \geqslant 1)$ are the same as in (2.8) from /6/. After determining $C_0, C_1, \ldots, C_N$ from (3.12), the solution of the IE (3.4) is given by (3.11) in the even case. The solution of the IE (3.4) in the odd case can be constructed similarly when $G(x) = \operatorname{sh} \varepsilon x$. Combining the even and odd solutions, the solution of the IE (3.4) can be written down in the case when the function $G(x)$ can be represented by a Fourier series.

The exact solution can be found by using the method elucidated above for problems a, b, c for $\alpha = 2\beta = 2\pi$, $L(u) = \operatorname{th} \pi u$.

4. We present a numerical example of the solution of a contact problem. We consider a plate in the form of a half-plane, simply supported along the boundary $\alpha = 2\beta = \pi$, $L(u) = \operatorname{cth} \pi u/2$, $B = 2/\pi$ (problem b).

As already noted, the problem is solved in closed form in this case.

By (1.1) let the dimensionless function $f(r) \equiv 1$; $k = 0.1$. Then $a_0 = -2.402$; $a_1 = -0.9227$; $a_2 = 0.6671$; $b_0 = -97.15$; $b_1 = 1.734$; $b_2 = -0.9762$; $c_0 = 12.98$; $c_1 = -1.298$ and $c_2 = -1.172$. Solving the system of linear Eqs.(2.12) for this case, we find that $\alpha_- = -1.586$, $\beta_- = -93.01$, $M_- = 29.50$. By using (2.6) and (2.9) we write down the distribution function of the desired contact forces in the form

$$\varphi(\rho) = -(1.692 + 74.04\rho^2 + 18.78\rho^4)\, g^{-3}(\rho) \tag{4.1}$$

It follows from expression (4.1) that the function $\varphi(\rho)$ retains its sign in the interval $0.1 < \rho < 1$, which indicates the impossibility of separating the inclusion from the plate. Integrating (4.1) with respect to ρ between 0.1 and 1, we evaluate the dimensionless force $P^* = Pb/D$. In the case under consideration here $P^* = 62.24$. The quantity $M_- = M/D$ has the meaning of a dimensionless moment applied to the inclusion.

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